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Topology and Kinematics of a Complex Rigid System

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THE coupled Euler equations of motion for multiple-part satellites present a problem in attitude-control studies because of the numerous coordinate transformations that have to be performed and that are not directly apparent when written in vector-dyadic form. The manner in which different parts of the system are identified has a decisive effect on the ease of reduction by a computer to scalar components as required in practical application. The purpose of this note is twofold. The first purpose is to give a topological classification of the various parts that exhibits in detail the physical interaction among the parts. This is of great importance in computer programing in that, given the identification for a particular part, the identification of all parts with which it interacts is obtained by a simple algebraic operation on the former's identification. Thus the computer has the capability of associating just those parts, relative to a given part, that are necessary to effect a series of coordinate transformations to the main body. The second purpose is to derive the necessary formulas that relate quantities and their time rates, as seen in a coordinate system attached to each part, to the values of these quantities relative to the reference frame of the main body. The formulas so obtained, in connection with the topology of each part, are given in a compact form that readily lends itself to the use of tensor notation and so programed on a computer. The text ends with a detailed example for the case of an unbalanced 2-gimbal gyro. The acceleration of the center of mass of the rotor is required with components on the main body. The motion of the rotor, however, is known only relative to the coordinate system of the inner gimbal, the inner gimbal relative to the outer gimbal, and so on.

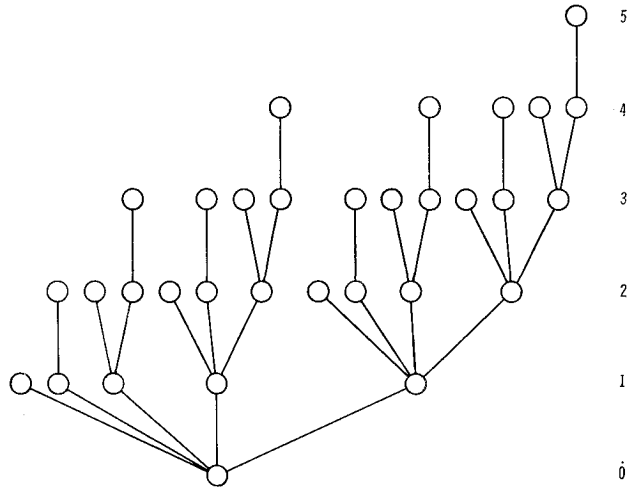

Fig. 1 Topology of interactions.

Received December 21, 1964; revision received April 1, 1965. This paper was prepared under the sponsorship of the Douglas Aircraft Company Independent Research and Development Program Account No. 80335-100.

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Topology

We shall consider a system of rigid bodies such that any two parts can interact only through relative rotation about a common fixed axis. The possibility of any closed-loop relationship between three or more parts will be excluded. It is felt that this is broad enough to include most real situations to be found in practice. Extremely complex interactions can occur, but they are all reducible to a simple topological classification of two dimensions. A convenient part will be isolated and called the main body, so that all other parts can be referred to this one by the notation to be developed. An interaction group is tentatively defined as a set of parts (elements) such that a given part of the total system either does or does not belong to the group depending upon whether that part can or cannot interact directly with some element of the group. If the main body interacts with each group, we have a closed set and, in fact, just one primitive interaction group (satellite), and so we modify the definition by excluding the main body as a member. The separation into disjoint groups provided by excluding the main body leads to identification of any rigid part of the system by its topological relationship to the main body. An element of a group is said to be removed by k if k is the smallest integer obtainable by counting from zero at the main body to the element in question through an unbroken chain of interconnected parts (Fig. 1),† the order of a group being next defined as the largest value of k obtained as one ranges over all elements of the group. Each part of the total system can now be specified by means of an ordered block of l characters $(j_1\ j_2\ \ldots\ j_l)$ in the manner suggested by Fig. 2. In general j_1 indicates the number of an l-order group, j_2 the number of an appropriate element that is removed by 2 from the main body in this group, j_3 the number of an element removed by 3 and attached to the previous element, etc. In case one is speaking of an element removed by k in an l-order group, the numbering would just be $j_1\ j_2\ \ldots\ j_k\ 0$

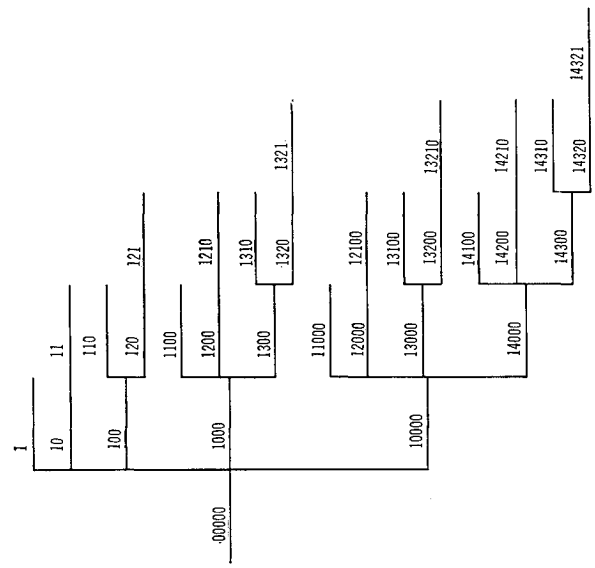

Fig. 2 Identification of elements.

† For example, the rotor of a 2-gimbal gyro is removed by 3, whereas an inertia wheel is only removed by 1.

. . . 0 with l-k zeros; the main body could thus be tagged, if desired, as 00 . . . 0 with l zeros for any l (it is the only part that does not have a unique label). This numbering program allows for any conceivably intricate relationship among the parts as initially stated.

Kinematics

At this point we consider the serial transformations of many variables of interest as implied in the multiple body equation[1]

$$\sum_{i=0}^{n} \frac{d}{dt}(\mathbf{I}^i \cdot \omega^i) + \sum_{i=0}^{m} m^i \mathbf{r}^i \times \frac{d^2}{dt^2}\mathbf{r}^i = \sum_{i=0}^{n} \mathbf{l}^i + \sum_{i=0}^{n} \mathbf{r}^i \times \mathbf{f}^i \tag{1}$$

For each part, $\mathbf{I}^i$ and $\mathbf{l}^i$ are the inertia dyadic and torque about its center of mass whereas ω^i, the angular velocity, is relative to space, and $\mathbf{r}^i$ is the displacement from the composite center of mass. The time derivatives are relative to space so that with respect to the main body ($i = 0$) these become

$$(d/dt)\mathbf{I}^i = \dot{\mathbf{I}}^i + \omega^0 \times \mathbf{I}^i - \mathbf{I}^i \times \omega^0 \tag{2a}$$

$$(d/dt)\boldsymbol{\omega}^i = \dot{\boldsymbol{\omega}}^i + \boldsymbol{\omega}^0 \times \boldsymbol{\omega}^i \tag{2b}$$

$$(d^2/dt^2)\mathbf{r}^i = \ddot{\mathbf{r}}^i + 2\boldsymbol{\omega}^0 \times \dot{\mathbf{r}}^i + \dot{\boldsymbol{\omega}}^0 \times \mathbf{r}^i + \boldsymbol{\omega}^0 \times (\boldsymbol{\omega}^0 \times \mathbf{r}^i) \tag{2c}$$

It is desired to know these quantities relative to main body coordinates. In general, $\mathbf{I}^i$ is given relative to the reference frame of the ith part whereas the primitive motion (rotation) of this part is known initially only with respect to the part below it in the chain; a translation of the center of mass of the ith part may also be involved if it is not on the common axis.

If the identification $j_1 j_2, \ldots, j_k 0, \ldots, 0$ is given for a part, it fixes that for each part below it on the chain. In fact, each one is identifiable relative to this part by its k. In what follows we will fix our attention on a particular $j_1 j_2, \ldots, j_k\, 0, \ldots, 0$. The superscript k will then be given to this part, $k - 1$ to the one below it, and so on. We should not confuse this notation with the original "i" in (1).

A digression on an important concept follows. The terms vector and dyadic will be used here with an added meaning of components relative to a basis implied. For example, consider an abstract vector quantity. Let this abstract quantity be resolved along some basis B^1 so that the components are $v_1{}^1, v_2{}^1, v_3{}^1$. In a similar way it is resolved along B^2 with components $v_1{}^2, v_2{}^2, v_3{}^2$. We can do this with any number of bases. Now, suppose we have some fundamental basis B^* (separate from any of the preceding) and do the following: in B^*, construct vectors $\mathbf{v}^1$, with components $v_1{}^1, v_2{}^1, v_3{}^1$; $\mathbf{v}^2$, with components $v_1{}^2, v_2{}^2, v_3{}^2$; and so on. It is obvious that $\mathbf{v}^1, \mathbf{v}^2, \ldots$, are all different if the bases $B^1, B^2, \ldots$ do not coincide. It then follows that $\mathbf{v}^i$ is changed into $\mathbf{v}^j$ by a dyadic $\mathbf{A}_j{}^i$ whose components in B^* are equal to the elements of the transformation matrix that takes B^i into B^j. Similarly, if we have an abstract dyadic quantity, the meaning of its representation $\mathbf{D}^1, \mathbf{D}^2, \ldots$, should be clear. The point of view that we have just developed makes the following analysis more compact than it would be with the usual interpretation. To justify this somewhat, we write an expression for the transformation of time derivatives between two bases B^i and B^j:

$$\dot{\mathbf{r}}^j = \mathbf{A}_j{}^i \cdot \dot{\mathbf{r}}^i + \boldsymbol{\omega}^j \times \mathbf{A}_j{}^i \cdot \mathbf{r}^i \tag{3}$$

where, for simplicity, $\boldsymbol{\omega}^j$ stands for the angular velocity of B^i about B^j with components in B^j. The terms $\dot{\mathbf{r}}^j$ and $\dot{\mathbf{r}}^i$ are just the time derivatives relative to B^*. It is obvious that the term $\boldsymbol{\omega}^i \times \mathbf{A}_j{}^i \cdot \mathbf{r}^j$ is more complex when written in terms of matrix operations.‡ Yet (3) is a relation between components in B^i and B^j but written in the more compact vector-dyadic language. Hence our lengthy discussion of this point of view.

To each part P^i is attached a coordinate basis B^i situated at the center of mass and making the moment of inertia diagonal.§ It is assumed that we know the displacement and angular velocity of P^i relative to P^{i-1} with components in B^{i-1}. All of this information will be represented by the vectors $\mathbf{b}^i$ and $\boldsymbol{\omega}^i$, respectively, with components understood. The moment of inertia dyadic of P^k about its own center of mass, having components in B^k and identified as such as $\mathbf{I}^k$, is also given. At a later time we will want to know this dyadic with components in B^0, and will there be known as $\mathbf{I}_0{}^k$. As a final preliminary we have the dyadic $\mathbf{A}^i$ that takes a unit vector with components in B^i into a unit vector with components in B^{i-1}.

The dyadic $\mathbf{A}_0{}^k$ that changes B^k to B^0 is simply written as the following product:

$$\mathbf{A}_0{}^k = \mathbf{A}^1 \cdot \mathbf{A}^2 \ldots \mathbf{A}^k = \mathbf{E} \cdot \mathbf{A}^1 \cdot \mathbf{A}^2 \ldots \mathbf{A}^k = \mathbf{A}^0 \cdot \mathbf{A}^1 \cdot \mathbf{A}^2 \ldots \mathbf{A}^k = \prod_{i=0}^{k} \mathbf{A}^i \tag{4}$$

where $\mathbf{E}$ is the idemfactor.

Now the distance from B^i to B^{i+1} has its components in B^0 given by $\mathbf{A}_0{}^i \cdot \mathbf{b}^{i+1}$. We can thus write the displacement of P^k from P^0 as a sum over all segments:

$$\mathbf{b}_0{}^k = \mathbf{b}^1 + \mathbf{A}_0{}^1 \cdot \mathbf{b}^2 + \mathbf{A}_0{}^2 \cdot \mathbf{b}^0 + \ldots + \mathbf{A}_0{}^{k-1} \cdot \mathbf{b}^k = \sum_{i=0}^{k-1} \mathbf{A}_0{}^i \cdot \mathbf{b}^{i+1}$$

where from (4) $\mathbf{A}_0{}^0 = \mathbf{E}$. By a parallel argument we get the angular velocity of P^k about P^0 with components in B^0

$$\boldsymbol{\omega}_0{}^k = \sum_{i=0}^{k-1} \mathbf{A}_0{}^i \cdot \boldsymbol{\omega}^{i+1}$$

with $\boldsymbol{\omega}_0{}^0 = \mathbf{o}$ ($\mathbf{o}$ is the zero or null vector). The first time derivative with respect to B^0 of the distance from B^i to B^{i+1} has its components in B^0 as in (3):

$$\dot{\mathbf{A}}_0{}^i \cdot \mathbf{b}^{i+1} + \boldsymbol{\omega}_0{}^i \times \mathbf{A}_0{}^i \cdot \mathbf{b}^{i+1}$$

so that the velocity of P^k relative to P^0, with components in B^0, is as in (5f). In a similar way one obtains an expression for the components in B^0 of the time derivative with respect to B^0 of the angular velocity of P^k about P^0 as (5g). The acceleration of P^k relative to P^0 with its components in B^0 is now seen directly to be (5i). The last relations needed are those for the moment of inertia, and these are given simply by (5j) and (5k):

$$\mathbf{A}_0{}^k = \prod_{i=0} \mathbf{A}^i \tag{5a}$$

$$\mathbf{A}_0{}^0 = \mathbf{E} \tag{5b}$$

$$\mathbf{b}_0{}^k = \sum_{i=0}^{k-1} \mathbf{A}_0{}^i \cdot \mathbf{b}^{i+1} \tag{5c}$$

$$\boldsymbol{\omega}_0{}^k = \sum_{i=0}^{k-1} \mathbf{A}_0{}^i \cdot \boldsymbol{\omega}^{i+1} \tag{5d}$$

$$\boldsymbol{\omega}_0{}^0 = \mathbf{o} \tag{5e}$$

$$\dot{\mathbf{b}}_0{}^k = \sum_{i=0}^{k-1} (\mathbf{A}_0{}^i \cdot \dot{\mathbf{b}}^{i+1} + \boldsymbol{\omega}_0{}^i \times \mathbf{A}_0{}^i \cdot \mathbf{b}^{i+1}) \tag{5f}$$

$$\dot{\boldsymbol{\omega}}_0{}^k = \sum_{i=0}^{k-1} (\mathbf{A}_0{}^i \cdot \dot{\boldsymbol{\omega}}^{i+1} + \boldsymbol{\omega}_0{}^i \times \mathbf{A}_0{}^i \cdot \boldsymbol{\omega}^{i+1}) \tag{5g}$$

$$\dot{\boldsymbol{\omega}}_0{}^0 = \mathbf{o} \tag{5h}$$

‡ In this latter language, the factor $\boldsymbol{\omega} \times$ is replaced by an antisymmetric matrix Ω.

§ The more general situation can be incorporated in a straightforward way and will not be dealt with here.

$$\ddot{\mathbf{b}}_0{}^k = \sum_{i=0}^{k=1} [\mathbf{A}_0{}^i \cdot \ddot{\mathbf{b}}^{i+1} + 2\boldsymbol{\omega}_0{}^i \times \mathbf{A}_0{}^i \cdot \dot{\mathbf{b}}^{i+1} + \dot{\boldsymbol{\omega}}_0{}^i \times \mathbf{A}_0{}^i \cdot \mathbf{b}^{i+1} + \boldsymbol{\omega}_0{}^i \times (\boldsymbol{\omega}_0{}^i \times \mathbf{A}_0{}^i \cdot \mathbf{b}^{i+1})] \quad (5i)$$

$$\mathbf{I}_0{}^k = \mathbf{A}_0{}^k \cdot \mathbf{I}^k \cdot {}^T\mathbf{A}_0{}^k \quad (5j)$$

$$\dot{\mathbf{I}}_0{}^k = \boldsymbol{\omega}_0{}^k \times \mathbf{A}_0{}^k \cdot \mathbf{I}^k \cdot {}^T\mathbf{A}_0{}^k - \mathbf{A}_0{}^k \cdot \mathbf{I}^k \cdot {}^T\mathbf{A}_0{}^k \times \boldsymbol{\omega}_0{}^k \quad (5k)$$

with ${}^T\mathbf{A}_0{}^k$ being the transpose or conjugate of $\mathbf{A}_0{}^k$.

Example

We will assume a satellite P^0 on which is mounted an unbalanced 2-gimbal gyro with P^1 being the outer gimbal, P^2 the inner gimbal, and P^3 the rotor. A coordinate system $B^i (i = 0, 1, 2, 3)$ is attached rigidly to P^i at its center of mass. The dyadic $\mathbf{A}^i$ takes components in B^i to components in B^{i-1}; $\mathbf{A}^0 = \mathbf{E}$. The center of mass of P^i relative to B^{i-1} is $\mathbf{b}^i$ (the components are understood to be in B^{i-1} by our convention). The rotation of B^i $(i = 1, 2, 3)$ relative to B^{i-1}, with components known in B^{i-1}, is $\boldsymbol{\omega}^i$. It is desired to know the acceleration $\ddot{\mathbf{b}}_0{}^3$ of P^3 relative to P^0, with components in B^0, in terms only of the known quantities just given. By application of formula (5i), in conjunction with (5a, 5d, and 5g), we have

$$\begin{aligned}\ddot{\mathbf{b}}_0{}^3 = {} & \ddot{\mathbf{b}}^1 + \mathbf{A}^1 \cdot \ddot{\mathbf{b}}^2 + 2\boldsymbol{\omega}^1 \times \mathbf{A}^1 \cdot \dot{\mathbf{b}}^2 + \dot{\boldsymbol{\omega}}^1 \times \mathbf{A}^1 \cdot \mathbf{b}^2 + \\ & \boldsymbol{\omega}^1 \times (\boldsymbol{\omega}^1 \times \mathbf{A}^1 \cdot \mathbf{b}^2) + \mathbf{A}^1 \cdot \mathbf{A}^2 \cdot \ddot{\mathbf{b}}^3 + \\ & 2(\boldsymbol{\omega}^1 + \mathbf{A}^1 \cdot \boldsymbol{\omega}^2) \times \mathbf{A}^1 \cdot \mathbf{A}^2 \cdot \dot{\mathbf{b}}^3 + \\ & (\mathbf{A}^1 \cdot \dot{\boldsymbol{\omega}}^2 + \boldsymbol{\omega}^1 \times \mathbf{A}^1 \cdot \boldsymbol{\omega}^2) \times \mathbf{A}^1 \cdot \mathbf{A}^2 \cdot \mathbf{b}^3 + \\ & (\boldsymbol{\omega}^1 + \mathbf{A}^1 \cdot \boldsymbol{\omega}^2) \times [(\boldsymbol{\omega}^1 + \mathbf{A}^1 \cdot \boldsymbol{\omega}^2) \times \mathbf{A}^1 \cdot \mathbf{A}^2 \cdot \mathbf{b}^3] \end{aligned} \quad (6)$$

We have in (6) a general relation but, as only relative rotation can occur, we can write some of the foregoing terms more explicitly. For example, let B^i rotate with respect to B^{i-1} about an axis $\hat{\mathbf{n}}^i$, with components in B^{i-1}, through an angle ϕ^i. Then $\boldsymbol{\omega}^i = \hat{\mathbf{n}}^i \dot{\phi}^i$ and $\dot{\boldsymbol{\omega}}^i = \hat{\mathbf{n}}^i \ddot{\phi}^i$ (time differentiation is with respect to B^{i-1}). Also, let $\mathbf{b}^i$ be made up of two parts; these are a displacement from B^{i-1} to that point on $\hat{\mathbf{n}}^i$ where the plane normal to $\hat{\mathbf{n}}^i$ passes through B^i (called $\mathbf{a}^i$) plus a displacement from this point to B^i (called $\mathbf{c}^i$). Thus $\dot{\mathbf{b}}^i = \boldsymbol{\omega}^i \times \mathbf{c}^i = \dot{\phi}^i \hat{\mathbf{n}}^i \times \mathbf{c}^i$ and $\ddot{\mathbf{b}}^i = \dot{\boldsymbol{\omega}}^i \times \mathbf{c}^i + \boldsymbol{\omega}^i \times (\boldsymbol{\omega}^i \times \mathbf{c}^i) = \ddot{\phi}^i \hat{\mathbf{n}}^i \times \mathbf{c}^i + (\dot{\phi}^i)^2 \hat{\mathbf{n}}^i \times (\hat{\mathbf{n}}^i \times \mathbf{c}^i)$. At this point, on substitution into (6), we have all of the information on the right-hand side known as it would be in practice.

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NERVA Thermal and Fluid-Flow Analysis

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Introduction

This note discusses a digital computer program developed to accomplish over-all performance studies with respect to the thermal and fluid-flow analysis in the nuclear engine for rocket vehicle application (NERVA) program. In evaluating the performance of a system, the interactions between the component characteristics and the system behavior depends on the steady state behavior of the system and upon five major transient effects. These transient effects can be described by thermal, nuclear, pneumatic, acoustic, and mechanical response characteristics, and all of them need to be considered at some time in an analysis program. Limitations of time and money preclude the consideration of all of the pertinent effects in one analysis program for a complicated system such as NERVA and have led to the establishment of program analysis requirements based on separation of effects. Since acoustical and mechanical transients normally involve frequencies much higher than 1 cps, they would require much more computer time for complicated systems than slower thermal, nuclear, and pneumatic transients. Although the provision for nuclear transients is included in the computer program, this discussion will be limited to the thermal and pneumatic considerations; control-system characteristics are also excluded.

Heat-Transfer and Fluid-Flow Data

Early in the NERVA program it was recognized that data representing the thermal and hydraulic behavior of hydrogen were meager and needed to be supplemented before realistic use of a program, such as described in this note, could be made. Studies performed during the past several years, and proceeding on a continuing basis, have provided this information to a large degree. The correlations used in this program are of necessity quite flexible and are based on data from various sources.

The coolant heat-transfer correlation used is based on data[1,2] that formally separates the regions of "liquid," two-phase flow, and "gas," although in the supercritical region there is actually no distinction between the regions. The exhaust gas heat-transfer correlation is based on scale-model tests for nozzles of similar geometry.[3,4]

In nuclear rocket systems of the NERVA type, thermal radiation from the core to the surrounding surfaces is appreciable and is accounted for by suitable shape factors.[5] The nuclear heating aspects in the system are accounted for by approximations of more exact calculations.[6]

General Program Description

The system being simulated is described primarily in terms of thermal nodes, fluid-flow elements, and other elements. Each thermal node is specified in terms of the nature and amount of the solid material assumed to be centered at the thermal node and its share of the nuclear and gamma radiation-flux levels. Its current state is specified by the temperature. The initial temperature must be supplied for every thermal node.

In a given fluid element, the inlet and outlet cross-sectional areas for flow may be different, and the corresponding momentum change is allowed for.[7] The effective area for pressure-drop and heat-transfer coefficient calculations may differ from both of the other areas for that element. Additional loss coefficients are used to allow for classical friction pressure drops, surface roughness factor, entrance or exit losses, or valve characteristics.

The capacitance of the flow element is expressed as the volume available for storage of fluid, and this volume is assumed to contain fluid at the average of the inlet and outlet densities. The length of the flow element is specified along with the distance from the entrance L_e, to allow for boundary-layer growth and entrance effects. The tube can be representative of n similar tubes of identical geometry and conditions, and the term flow element applies to the group. Analytical flexibility in dealing with complex geometrical shapes has been obtained in the program, and coolant passages can be arranged in various series and/or parallel combinations.

Presented as Preprint 64-392 at the 1st AIAA Annual Meeting, Washington, D. C., June 29–July 2, 1964; revision received May 14, 1965. The authors wish to acknowledge the work of L. Felts in programing for the digital computer and the aid of D. S Adamson with the engineering analysis.

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